INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 18, 1009–1019 (1994)

FINITE ELEMENT IMPLEMENTATION OF BOUNDARY CONDITIONS FOR THE PRESSURE POISSON EQUATION OF INCOMPRESSIBLE FLOW

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SUMMARY

In this paper we address the problem of the implementation of boundary conditions for the derived pressure Poisson equation of incompressible flow. It is shown that the *direct* Galerkin finite element formulation of the pressure Poisson equation automatically satisfies the inhomogeneous Neumann boundary conditions, thus avoiding the difficulty in specifying boundary conditions for pressure. This ensures that only physically meaningful pressure boundary conditions consistent with the Navier–Stokes equations are imposed. Since second derivatives appear in this formulation, the conforming finite element method requires C^1 continuity. However, for many problems of practical interest (i.e. high Reynolds numbers) the second derivatives need not be included, thus allowing the use of more conventional C^0 elements. Numerical results using this approach for a wall-driven contained flow within a square cavity verify the validity of the approach. Although the results were obtained for a two-dimensional problem using the *p*-version of the finite element method, the approach presented here is general and remains valid for the conventional *h*-version as well as three-dimensional problems.

KEY WORDS Pressure Poisson equation Incompressible flow Finite element method

1. INTRODUCTION

Finite element solutions of transient incompressible viscous flow in terms of the primitive variables have typically followed two distinct approaches.

One approach has been to discretize the time-dependent Navier–Stokes equations in their original primitive variable form (momentum equations and a continuity equation).[1–3] The continuity equation can be considered as a constraint on the divergence of the velocity rather than as a fully fledged equation coupling the pressure to the velocity. Thus the pressure in these equations appears as a Lagrange multiplier that ensures the satisfaction of the solenoidal constraint. Moreover, the interpolating basis functions for pressure must be at least one order less than those for velocity components.[4]

An independent concern with the discretization of the equations in their original form is that the resulting discrete (matrix) operator is neither symmetric nor positive definite.[5] This is not

CCC 0271–2091/94/111009–11

Received 21 February 1989
Revised 8 November 1993

important when direct methods such as those based on Gaussian elimination are used to solve the resulting system of equations. Direct methods, however, can be prohibitively expensive in terms of both computer storage and time for large three-dimensional problems. For this reason there is serious interest in methods such as preconditioned iterative methods that use less storage and time to solve the resulting equations. The non-symmetric and indefinite discrete operator obtained with the original Navier-Stokes equations often causes most iterative techniques to diverge or to converge very slowly.

An alternative to the approaches that solve the primitive equations in their original form is to replace the continuity equation with a Poisson equation for the pressure, thus decoupling the momentum and continuity equations except on the domain boundary. This approach has been widely used in finite difference solutions of incompressible flow problems (see Reference 6 for a review of this approach). However, it has not been used as extensively in finite element formulations.[4,7] One reason is that there are theoretical and practical difficulties connected with the specification of boundary conditions for the pressure Poisson equation. The issue of the boundary conditions for the derived Poisson equation for the pressure has been a topic of intense debate and discussion in recent years.[8–10] In general there are no *a priori* boundary conditions on the pressure. Moreover, there are difficulties with the *direct* implementation of the derived boundary conditions for the pressure, particularly in finite difference methods.[8,11–13]

We will show in the following sections that the difficulties in specifying the pressure boundary conditions can be circumvented by utilizing properties inherent in the finite element formulation. Namely, we need not specify the inhomogeneous Neumann pressure boundary conditions. In subsequent sections we present results for test problems to demonstrate the validity of the approach.

2. GOVERNING EQUATIONS

The governing equations for the unsteady, isothermal flow of incompressible, viscous, Newtonian fluids in a domain $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$), with no body forces and constant properties, are

$$\partial \mathbf{u}/\partial t + \nabla \cdot (\mathbf{u}\,\mathbf{u}) - \nu \nabla^2 \mathbf{u} + \nabla p = \mathbf{0}, \tag{1a}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1b}$$

subject to initial conditions

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad (\nabla \cdot \mathbf{u}_0 = 0) \quad \text{in } \bar{\Omega} \equiv \Omega \cup \Gamma \tag{2a}$$

and boundary conditions

$$\mathbf{u} = \mathbf{w}(\mathbf{x}, t) \quad \left(\int_\Gamma \mathbf{w} \cdot \mathbf{n}\, d\Gamma = 0,\ \mathbf{n} \cdot \mathbf{u}_0(\mathbf{x}) = \mathbf{n} \cdot \mathbf{w}(\mathbf{x}, 0) \right) \quad \text{on } \Gamma \equiv \partial\Omega, \tag{2b}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity, $p(\mathbf{x}, t)$ is the kinematic pressure (pressure divided by density), ν is the kinematic viscosity ($\nu \geqslant 0$), $\mathbf{w}(\mathbf{x}, t)$ is a given function on Γ and $\mathbf{n}$ denotes the outward normal unit vector at Γ.

The system (1) is equivalent to[10]

$$\mathbf{a} + \nabla p = \mathbf{f}(\mathbf{u}) \quad \text{in } \Omega, \qquad \mathbf{a} = \dot{\mathbf{w}} \quad \text{on } \Gamma, \tag{3a}$$

and

$$\nabla^2 p = \nabla \cdot (\mathbf{f} - \mathbf{a}) \quad \text{in } \Omega, \qquad \partial p/\partial n = \mathbf{n} \cdot (\mathbf{f} - \dot{\mathbf{w}}) \quad \text{on } \Gamma, \tag{3b}$$

where $\mathbf{a} \equiv \partial \mathbf{u}/\partial t$, $\mathbf{f}(\mathbf{u}) = \nu \nabla^2 \mathbf{u} - \nabla \cdot (\mathbf{u}\,\mathbf{u})$ and the pressure Poisson equation has been obtained by taking the divergence of the momentum equation (1a). Note that the pressure Poisson equation can be obtained in various equivalent forms.[6,10,14]

3. FINITE ELEMENT FORMULATIONS

The finite element solution of the continuum equations (3) is obtained by establishing a weak form of the equations using the Galerkin procedure.[2] This formulation, however, may be achieved via different routes.

Gresho and his colleagues at the Lawrence Livermore National Laboratory have over the past several years developed a finite element formulation which is analogous to the derivation of the continuum PPE.[10,15] In this approach, instead of directly constructing the discrete Laplacian operator from equation (3b), they derive the discrete analogue of (3b) which they call the *consistent* discretized pressure Poisson equation. One advantage of such an approach, among others, is that no pressure boundary conditions need be imposed to generate the *consistenct discretized Laplacian* operator, since they are always built in automatically.[10]

Alternatively, the finite element formulation of the pressure Poisson equation can be achieved by *direct* construction of the discrete Laplacian of equation (3b). The advantage of this approach, in addition to being much simpler, is that it allows equal-order interpolation for velocity and pressure. The main drawback, however, is that in this scheme the discrete operator must be generated with the natural pressure boundary condition. Earlier *direct* discretizations of the PPE have either obtained the normal derivative of the pressure on the boundary by applying the normal component of the momentum equations at the boundary[4,16] or assumed homogenous Neumann boundary conditions.[17] While these assumptions give good results and are well justified in many instances, they can result in poor approximations of the flow field in certain circumstances. (For an interesting analysis and comparison of the *consistent* and *direct* approaches see Reference 18.)

The difficulty in the implementation of the natural boundary condition for the pressure, however, can be circumvented by judicious application of the finite element procedure to the pressure Poisson equation. Applying the standard Galerkin-weighted residual method to equation (3b), we arrive at the equivalent integral equation

$$\int_\Omega \phi \nabla^2 p \, d\Omega = \int_\Omega \phi [\nabla \cdot (\mathbf{f} - \mathbf{a})] \, d\Omega, \tag{4}$$

where ϕ is a set of linearly independent basis functions chosen such that all global boundary conditions are satisfied.[2,19] Applying the Green–Gauss theorem or integrating by parts, we obtain

$$\int_\Omega \phi \nabla^2 p \, d\Omega = \int_\Omega \nabla \cdot (\phi \nabla p) \, d\Omega - \int_\Omega \nabla \phi \cdot \nabla p \, d\Omega = \int_\Omega \nabla \cdot [\phi(\mathbf{f} - \mathbf{a})] \, d\Omega - \int_\Omega \nabla \phi \cdot (\mathbf{f} - \mathbf{a}) \, d\Omega, \tag{5a}$$

$$\int_\Gamma \phi \mathbf{n} \cdot \nabla p \, d\Gamma - \int_\Omega \nabla \phi \cdot \nabla p \, d\Omega = \int_\Gamma \phi \mathbf{n} \cdot (\mathbf{f} - \mathbf{a}) \, d\Gamma - \int_\Omega \nabla \phi \cdot (\mathbf{f} - \mathbf{a}) \, d\Omega \tag{5b}$$

or

$$\int_\Omega \nabla\phi \cdot \nabla p \, d\Omega = \int_\Omega [\nabla\phi \cdot (\mathbf{f} - \mathbf{a})] \, d\Omega + \int_\Gamma \phi \mathbf{n} \cdot [\nabla p - (\mathbf{f} - \mathbf{a})] \, d\Gamma. \tag{5c}$$

However, $\nabla p - (\mathbf{f} - \mathbf{a}) = \mathbf{0}$, to give finally

$$\int_\Omega \nabla\phi \cdot \nabla p \, d\Omega = \int_\Omega [\nabla\phi \cdot (\mathbf{f} - \mathbf{a})] \, d\Omega. \tag{6}$$

Remarks

1. Equation (6) provides a *weak* form of the pressure Poisson equation that automatically satisfies the inhomogenous Neumann boundary conditions. Thus the difficulty of the implementation of the pressure boundary conditions is circumvented in its entirety.
2. This formulation does not need to use $\nabla \cdot \mathbf{u} = 0$, i.e. it does not satisfy any discrete approximation to the solenoidal condition. The incompressibility condition is asymptotically satisfied by the convergent solution. However, as will be demonstrated below by numerical examples, for a reasonable discretization the mass conservation is acceptable.
3. All boundary conditions in this formulation are assumed not to vary with time.

4. SPATIAL AND TIME DISCRETIZATION

The two-dimensional components of velocity and pressure are discretized using the p-version of the finite element method with hierarchic basis functions. The use of hierarchic basis functions in finite element formulations has received increasing acceptance in recent years as an effective adaptive technique for discretizing partial differential operators.[20–22] The basic idea of the p-version is to use higher-order polynomials of degree p in regions where higher accuracy is desired, instead of reducing the mesh size, i.e. reducing the diameter h of the largest element (h-version). The element is hierarchic in the sense that the elemental discretized operator matrix of degree $p + 1$ contains as a submatrix the elemental matrix of degree p. This property is particularly useful when iterative solution techniques are employed. Moreover, the use of hierarchic basis functions provides a means to develop error indicators that indicate areas where a refinement gives the best improvement. Patera and his colleagues at Massachusetts Institute of Technology have developed a *spectral element* method for fluid dynamics which is similar to the p-version finite element method.[23,24] However, the difference is that hierarchic basis functions are not employed in the spectral element method.

The discretization of the weak form of the momentum equation (1a) and the pressure equation (6) leads to the coupled discrete equations

$$\mathbf{M}\,\dot{\mathbf{U}} = \mathbf{L}_A\,\mathbf{U} + \mathbf{L}_V\,\mathbf{U} + \mathbf{C}\,\mathbf{P} + \mathbf{R}, \tag{7a}$$

$$\mathbf{K}\,\mathbf{P} = \mathbf{L}_f\,\mathbf{U} + \mathbf{D}\,\dot{\mathbf{U}}, \tag{7b}$$

where $\mathbf{U}$ and $\mathbf{P}$ represent the elemental vectors of $\mathbf{u}$ and p respectively, $\mathbf{R}$ is the velocity load vector to be determined from the boundary conditions (2), and the discretized matrices for a

single element are

$$M_e \equiv \int_{\Omega^e} \phi\phi^{\mathrm{T}}\, \mathrm{d}\Omega, \tag{7c}$$

$$L_{A_e} \equiv -\int_{\Omega^e} \phi \mathbf{u} \cdot \nabla\phi^{\mathrm{T}}\, \mathrm{d}\Omega, \tag{7d}$$

$$L_{V_e} \equiv -\int_{\Omega^e} \nu\nabla\phi \cdot \nabla\phi^{\mathrm{T}}\, \mathrm{d}\Omega, \tag{7e}$$

$$C_e \equiv -\int_{\Omega^e} \phi\nabla\phi^{\mathrm{T}}\, \mathrm{d}\Omega, \tag{7f}$$

$$K_e \equiv \int_{\Omega^e} \nabla\phi \cdot \nabla\phi^{\mathrm{T}}\, \mathrm{d}\Omega, \tag{7g}$$

$$L_{f_e} \equiv \int_{\Omega^e} \nabla\phi \cdot (\nu\nabla^2\phi^{\mathrm{T}} - \mathbf{u} \cdot \nabla\phi^{\mathrm{T}})\, \mathrm{d}\Omega, \tag{7h}$$

$$D_e = C_e^{\mathrm{T}}. \tag{7i}$$

The superscript T denotes matrix transposition.

Remarks

1. Since second derivatives (in the viscous part) appear in the *source* terms for the pressure equation (equation (7h)), the conforming finite element method requires C^1 continuity, i.e. both the velocity and its first derivatives must be continuous across the common boundary of adjoining elements so that the second derivatives are integrable. Such elements have been formulated and successfully used in a wide range of scientific and engineering problems.[25–30]
2. The use of C^1 elements generally involves more degrees of freedom per element than the standard C^0 basis functions. Also, it is well known that in the case of C^1 continuity, conformity is not enforced as easily as in the case of C^0 continuity where only the velocity needs to be continuous.[25] These computational drawbacks, however, may be overcome either by using techniques such as the discontinuous Galerkin method[31] or by dropping the viscous part from the integral on the right-hand side of equation (7h). However, as pointed out by Gresho,[32] in limiting cases such as steady Stokes flow the neglect of the viscous term yields a PPE with no driving force. Although theoretically unproven, our experience indicates that for low to moderate Reynolds numbers, the unsteady term can in certain cases compensate for the viscous term. In other words, the steady solution is attained by taking the $t \to \infty$ limit of a transient solution. However, in general, dropping the viscous term limits the applicability of the scheme to high Reynolds numbers. For high Reynolds numbers, i.e. convection-dominated flows, the viscous term is not as important as the advective term. Previous studies have also reached similar conclusions.[4] Thus for many problems of practical interest (i.e. high Reynolds numbers) the second derivatives need not be included in the numerical formulation, allowing the use of more conventional C^0 elements. In the numerical examples to follow, this approach has been adopted.

Therefore the problem reduces to the transient-iterative solution of two coupled discrete systems: a non-linear system for the velocity with the pressure as the source term (equation (7a)) and a linear system for the pressure with the velocity as the source term (equation (7b)).

The time integration of the momentum equations can be achieved by a fractional time-stepping (splitting) method.[33] This fractional scheme was first used by Orszag and Kells[34] and later developed by other investigators.[35–38]

The time-differencing scheme can be split into three steps

$$\mathbf{U}^1 = \mathbf{U}^n + \Delta t \mathbf{M}^{-1}\mathbf{L}_A\mathbf{U}^n, \tag{8a}$$

$$\mathbf{U}^2 = \mathbf{U}^1 + \Delta t \mathbf{M}^{-1}\mathbf{L}_V\mathbf{U}^2, \tag{8b}$$

$$\mathbf{U}^{n+1} = \mathbf{U}^2 + \Delta t \mathbf{M}^{-1}\mathbf{C}\mathbf{P}^n, \tag{8c}$$

where $\mathbf{U}^1$ and $\mathbf{U}^2$ are auxiliary velocity vectors, $\mathbf{U}^n$ and $\mathbf{U}^{n+1}$ are the velocity vectors at time steps n and $n+1$ respectively and the pressure $\mathbf{P}$ is computed from equation (7b). The boundary conditions used in each step of the fractional splitting method are the appropriate conditions for the corresponding equation being solved.[34–38] In the finite element method these conditions are incorporated in the formulation.

For the lid-driven square cavity problem the algorithmic expression of the splitting scheme is as follows.

0. Given $\mathbf{U}^n$.
1. Compute $\mathbf{P}^n$ from equation (7b) by setting $\mathbf{D}\mathbf{U}^{n+1} = 0$ on the right-hand side.
2. Update $\mathbf{U}^1$ from equation (8a).
3. Update $\mathbf{U}^2$ from equation (8b).
4. Update $\mathbf{U}^{n+1}$ from equation (8c).
5. Increase n to $n+1$ and go to step 1. Stop when a convergent solution is attained.

Note that only the advective operator in step 2, treated explicitly, imposes stability conditions for the scheme (Courant–Friedrich–Lewy condition). The viscous part (step 3) is solved implicitly and is thus unconditionally stable.

Although in general splitting offers considerable simplifications compared with implicit coupled schemes, it may suffer from larger time-stepping errors. However, it is possible to reduce the error of the fractional time-stepping method by applying higher-order local extrapolation techniques, e.g. Richardson extrapolation.[14] Another method, based on a Green function technique, was also proposed by Marcus[35] to reduce the time-splitting error.

5. NUMERICAL EXAMPLES

The approach presented above has been tested on the classical problem of the wall-driven contained flow within a unit square cavity. As suggested by Gresho,[32] we first consider a simple case, namely steady Stokes flow. As mentioned above, the steady state solution is obtained through time marching of our unsteady scheme. In fact, the unsteady term is the only driving force for the PPE in this example. However, we should stress that this may not work in all circumstances.

At $t = 0$ a constant velocity distribution is imposed at the lid. The pressure is calculated from the pressure equation (7b). With the known $(\mathbf{u}_0, p_0)$ at the initial step the computation marches forward to the first step. The discretized equations have been integrated forward in time using the fractional time-stepping (splitting) method (equations (8)). In this scheme the advective and

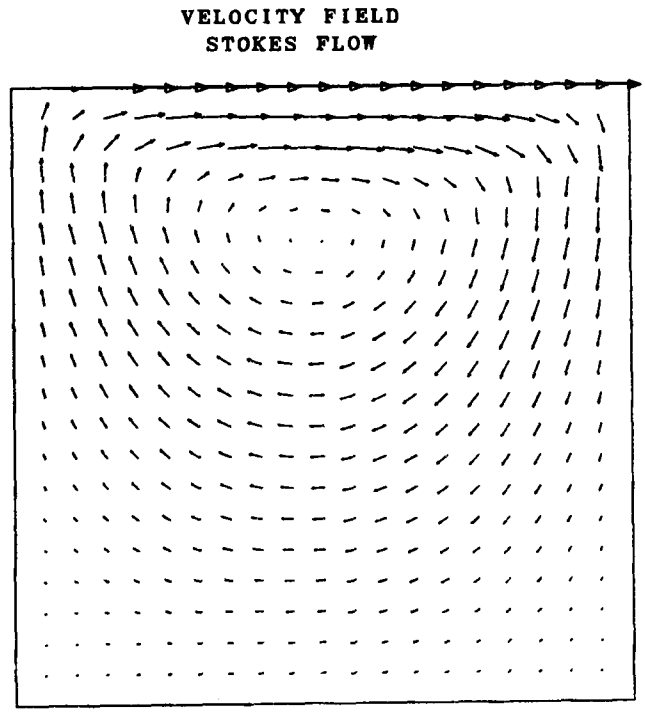

Figure 1. Velocity field in square cavity for Stokes flow

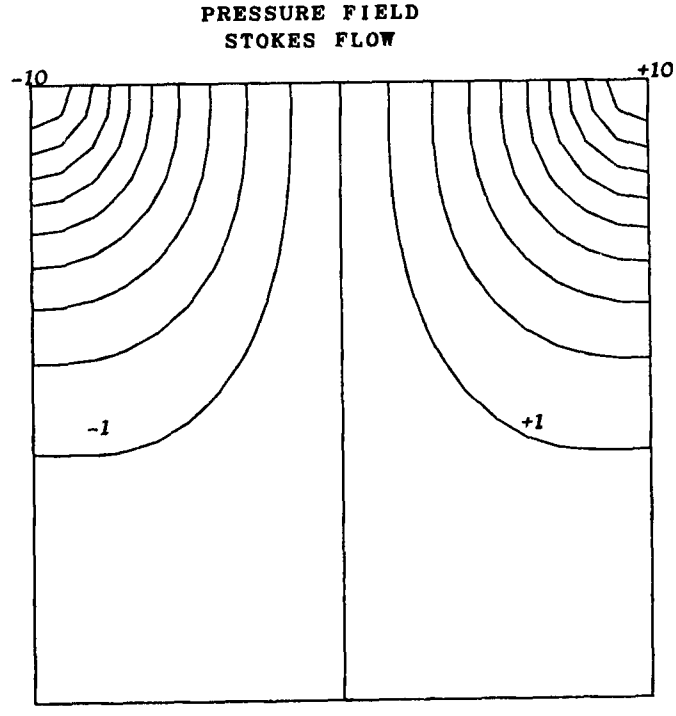

Figure 2. Pressure isobars for Stokes flow

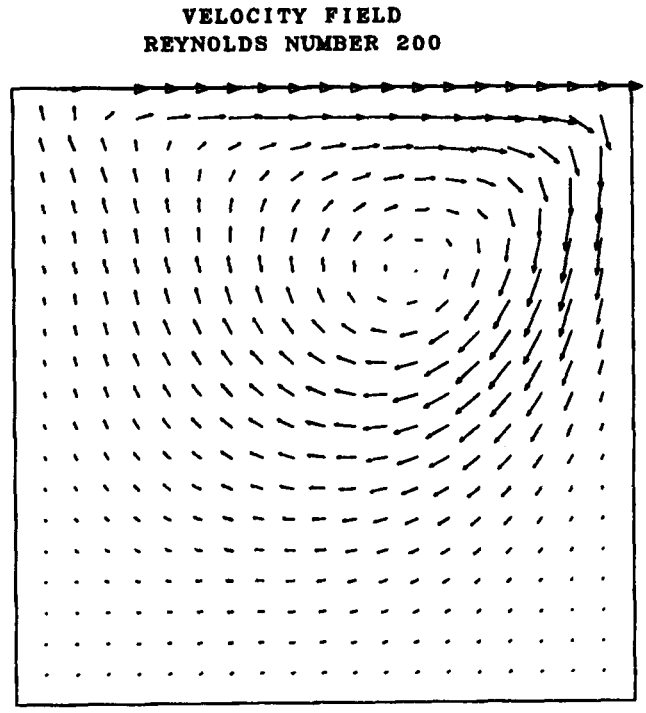

Figure 3. Velocity field in square cavity for Reynolds number of 200

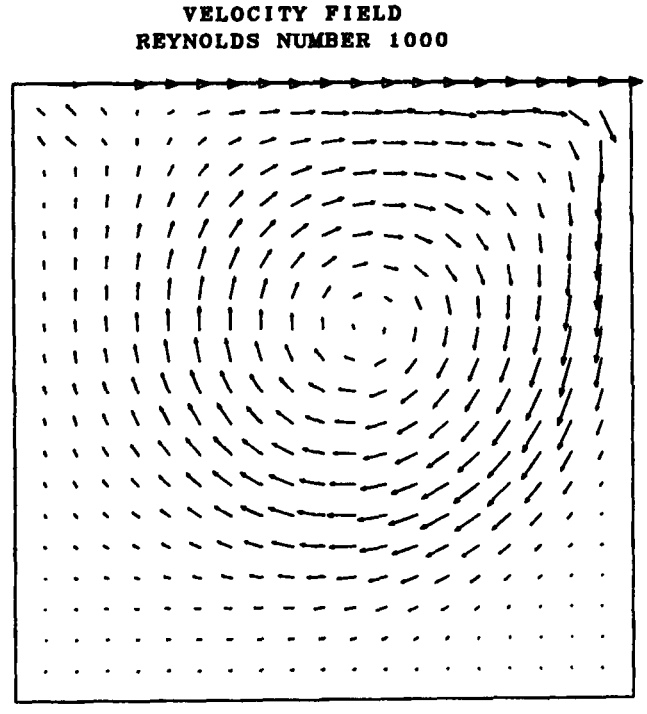

Figure 4. Velocity field in square cavity for Reynolds number of 1000

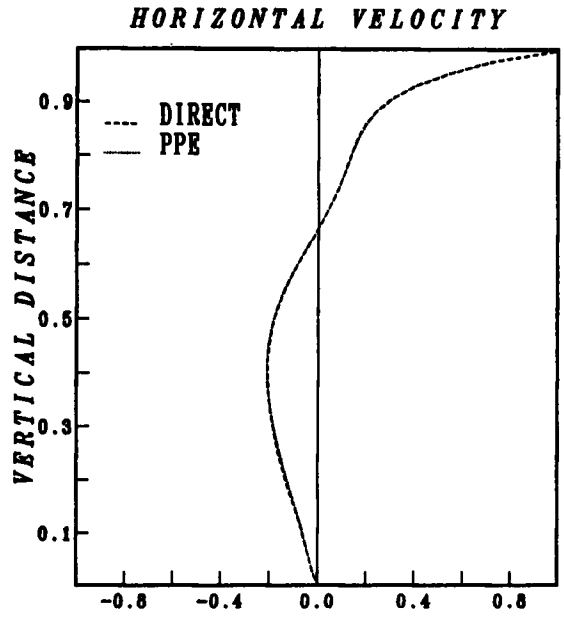

Figure 5. Horizonal velocity along centreline for direct and PPE solutions for Reynolds number of 200

final velocity steps (equations (8a) and (8c)) are solved using the preconditioned conjugate gradient method and the remaining steps (viscous and pressure parts) are solved using the conjugate gradient technique.

One hundred elements have been used in all cases with hierarchic polynomial basis functions of degree three. The no-slip boundary condition ($u = v = 0$) has been imposed at all boundaries except $y = 1$. To avoid the problem of singularities at the upper corners, we specify a velocity distribution at the lid which is zero at the upper two corners of the cavity but increases linearly to a constant value at the lid within the corner elements. The velocity and pressure fields are shown in Figures 1 and 2 respectively. These results are in good agreement with previous results from finite difference[39,40] and finite element[4,41] simulations.

We then considered the driven cavity problem for higher Reynolds numbers. The results for Reynolds numbers of 200 and 1000 are presented in Figures 3 and 4 respectively. These results are also in good agreement with previous results.

To confirm the validity of our approach, we also compared the numerical results presented above against direct solutions of the original Navier–Stokes equations (where no PPE is formed). The steady momentum and continuity equations were discretized using the p-version of the finite element method with hierarchic basis functions of order 10 for the velocity and two orders less for the pressure. The discretized equations were solved directly using Newton's method. Figure 5 displays the horizontal velocity along the centreline for a Reynolds number of 200 for both the direct and the pressure Poisson equation approaches. As this figure shows, the results are nearly identical.

6. CONCLUDING REMARKS

A finite element approach for the implementation of the boundary conditions for the pressure Poisson equation of incompressible flow is presented. In this approach a standard Galerkin-weighted residual method is applied to the time-dependent Navier–Stokes (momentum) equa-

tions and the Poisson equation for the pressure. It is shown that the *direct* Galerkin finite element formulation of the pressure Poisson equation automatically satisfies the inhomogenous Neumann boundary conditions for the pressure. Since second derivatives (in the viscous term) appear in this formulation, it requires C^1 continuity. However, this restriction may be relaxed for most practical cases of interest (i.e. high Reynolds numbers) by dropping the second-derivative terms. Numerical results for a wall-driven contained flow within a square cavity indicate that the approach is numerically stable and accurate.

Although the numerical results presented in this paper are obtained for the two-dimensional problem using the *p*-version of the finite element method, the approach is also valid for the conventional *h*-version as well as three-dimensional problems.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. P. M. Gresho for providing a number of valuable and insightful comments. They would also like to express their appreciation to an anonymous reviewer for critically reading the original manuscript.

REFERENCES

1. M. Kawahara, N. Yoshimura, K. Nakagawa and H. Oheaka, 'Steady and unsteady finite element anlaysis of incompressible viscous fluid', *Int. j. numer. methods eng.*, **10**, 437 (1976).
2. T. J. Chung, *Finite Element Analysis in Fluid Dynamics*, McGraw-Hill, New York, 1978.
3. K. J. Bathe and J. Dong, 'Solutions of incompressible viscous fluid flow with heat transfer using ADINA-F', *Comput. Struct.*, **26**, 17–31 (1987).
4. S. Y. Tuann and M. D. Olson, 'A transient finite element solution method for the Navier–Stokes equations', *Comput. Fluids*, **6**, 141–152 (1978).
5. M. Fortin, 'Old and new finite elements for incompressible flows', *Int. j. numer. methods fluids*, **1**, 347–364 (1981).
6. P. J. Roache, *Computational Fluid Dynamics*, Hermosa, Albuquerque, NM, 1976.
7. R. Glowinski and O. Pironneau, 'On a mixed finite element approximation of the Stokes problem', *Numer. Math.*, **33**, 397–424 (1979).
8. J. C. Strikwerda, 'Finite difference methods for the Stokes and Navier–Stokes equations', *SIAM J. Sci. Stat. Comput.*, **5**, 56–68 (1984).
9. P. J. Roache, 'A comment on the paper "Finite difference methods for the Stokes and Navier–Stokes equations" by J. C. Strikwerda', *Int. j. numer. methods fluids*, **8**, 1459–1463 (1988).
10. P. M. Gresho and R. L. Sani, 'On pressure boundary conditions for the incompressible Navier–Stokes equations', *Int. j. numer. methods fluids*, **7**, 1111–1145 (1987).
11. W. Briley, 'Numerical method for predicting three-dimensional steady viscous flow in ducts', *J. Comput. Phys.*, **14**, 8–28 (1974).
12. K. Ghia, W. Hanky and J. Hodge, 'Use of primitive variables in the solution of incompressible Navier–Stokes equations', *AAIA J.*, **17**, 298 (1979).
13. B. Alfrink, 'On the Neumann problem for the pressure in a Navier–Stokes model', *Proc. 2nd Int. Conf. on Numerical Methods in Laminar and Turbulent Flow*, Pineridge, Swansea, 1981, p. 389.
14. S. A. Orszag, M. Israeli and M. O. Deville, 'Boundary conditions for incompressible flows', *J. Sci. Comput.*, **1**, 75–111 (1986).
15. P. M. Gresho, S. T. Chan, R. L. Sani and C. D. Upson, 'A modified finite element method for solving the time-dependent incompressible Navier–Stokes equations, Part 1: Theory and Part 2: Applications', *Int. j. numer. methods fluids*, **4**, 557–640 (1984).
16. S. Abdallah and J. Dreyer, 'Dirichlet and Neumann boundary conditions for the pressure Poisson equation of incompressible flow', *Int. j. numer. methods fluids*, **8**, 1029–1036 (1988).
17. M. Kawahara and K. Ohmiya, 'Finite element analysis of density flow using the velocity correction method', *Int. j. numer. methods fluids*, **5**, 981 (1985).
18. P. M. Gresho, 'An analysis of the velocity correction method of M. Kawahara *et al.*', *Bull. Fac. Sci. Eng., Chuo Univ.*, **30**, 43–61 (1987).
19. O. C. Zienkiewicz, *The Finite Element Method*, McGraw-Hill, London, 1977.
20. B. Szabo, 'Selected references on the *p*-version of the finite element method', unpublished, 1988 (available on request from Washington University, St. Louis, MO).
21. O. C. Zienkiewicz, J. P. De S. R. Gago and D. W. Kelly, 'The hierarchical concept in finite element analysis', *Comput. Struct.*, **16**, 53–65 (1983).

22. I. Babuska, B. Szabo and I. N. Katz, 'The p-version of the finite element method', *SIAM J. Numer. Anal.*, **18**, 515–545 (1981).
23. A. T. Patera, 'A spectral element method for fluid dynamics: laminar flow in a channel expansion', *J. Comput. Phys.*, **54**, 468–488 (1984).
24. A. T. Patera, 'Advances and future directions of research on spectral methods', in A. K. Noor (ed.), *Computational Mechanics: Advances and Trends*, AMD Vol. 75, ASME, New York, 1987, pp. 411–427.
25. A. G. Peano, 'Hierarchies of conforming finite elements for plane elasticity and plate bending', *Comput. Math. Appl.*, **2**, 221–224 (1976).
26. A. G. Peano, 'Conforming approximations for Kirchhoff plates and shells', *Int. j. numer. methods eng.*, **14**, 1273–1291 (1979).
27. B. M. Irons, 'A conforming quartic triangular element for plate bending', *Int. j. numer. methods eng.*, **1**, 29–45 (1969).
28. J. Morgan and R. Scott, 'A nodal basis for C^1 piecewise polynomials of degree $n \geqslant 5$', *Math. Comput.*, **29**, 736–740 (1975).
29. I. N. Katz and D. W. Wang, 'The p-version of the finite element method for problems requiring C^1-continuity', *SIAM J. Numer. Anal.*, **22**, 1082–1106 (1985).
30. B. Q. Guo, 'The h–p version of the finite element method for elliptic equations of order $2m$', *Numer. Math.*, **53**, 199–224 (1988).
31. C. Johnson, *Numerical Solutions of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, Cambridge, 1987.
32. P. M. Gresho, personal communications, 1990.
33. N. N. Yanenko, *The Method of Fractional Steps*, Springer, New York, 1971.
34. S. A. Orszag and L. C. Kells, 'Transition to turbulence in plane Poiseuille and plane Couette flows', *J. Fluid Mech.*, **96**, 159 (1980).
35. P. S. Marcus, 'Simulation of Taylor–Couette flow, Part 1. Numerical methods and comparison with experiments', *J. Fluid Mech.*, **146**, 45 (1984).
36. S. A. Orszag and A. T. Patera, 'Secondary instability of wall-bounded shear flows', *J. Fluid Mech.*, **128**, 347 (1983).
37. K. Z. Korczak and A. T. Patera, 'An isoparametric spectral element method for solution of the Navier–Stokes equations in complex geometry', *J. Comput. Phys.*, **62**, 361 (1986).
38. E. Dean, R. Glowinski and C. H. Li, 'Supercomputer solutions of partial differential equation problems in computational fluid dynamics and in control', *Comput. Phys. Commun.*, **53**, 401–439 (1989).
39. O. R. Burggraf, 'Analytic and numerical studies of the structure of steady separated flows', *J. Fluid Mech.*, **24**, 113 (1966).
40. K. Gustafson and K. Halsi, 'Vortex dynamics of cavity flows', *J. Comput. Phys.*, **64**, 279–319 (1986).
41. M. D. Olson and S. Y. Tuann, 'New finite element results for the square cavity', *Comput. Fluids*, **7**, 123–135 (1979).